EXTREMAL DOMAINS FOR THE FIRST EIGENVALUE IN A GENERAL COMPACT RIEMANNIAN MANIFOLD

ERWANN DELAY AND PIERALBERTO SICBALDI

Abstract. We prove the existence of extremal domains with small prescribed volume for the first eigenvalue of the Laplace-Beltrami operator in any compact Riemannian manifold. This result generalizes a results of F. Pacard and the second author where the existence of a nondegenerate critical point of the scalar curvature of the Riemannian manifold was required.

Contents

1. Introduction and statement of the result

Let (M, g) be an n-dimensional Riemannian manifold, Ω a connected and open domain in M with smooth boundary, and $\lambda_\Omega > 0$ the first eigenvalue of the Laplace-Beltrami operator $-\Delta_g$ in Ω with zero Dirichlet boundary condition. The domain Ω is said to be *extremal* (for the first eigenvalue of the Laplace-Beltrami operator with zero Dirichlet boundary condition) if it is a critical point for the functional $\Omega \longmapsto \lambda_\Omega$ in the class of domains with the same volume.

An extremal domain is characterized by the fact that the first eigenfunction of the Laplace-Beltrami operator with zero Dirichlet boundary condition has constant Neumann data at the boundary. This result has been proved in the Euclidean space by P.R. Garabedian and M. Schiffer in 1953 [4], and in a general Riemannian manifold by A. El Soufi and S. Ilias in 2007 [2]. Extremal domains are then domains where the elliptic overdetermined problem

$$\begin{cases} \Delta_g u + \lambda\, u = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ g(\nabla u, \nu) = \text{constant} & \text{on } \partial\Omega \end{cases} \tag{1}$$

can be solved for some positive constant λ, where ν denotes the outward unit normal vector about $\partial\Omega$ for the metric g.

In $\mathbb{R}^n$ the only extremal domains are balls. This is a consequence of a very well known result by J. Serrin: if there exists a solution u to the overdetermined elliptic problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \langle \nabla u, \nu \rangle = \text{constant} & \text{on } \partial\Omega\,, \end{cases} \tag{2}$$

for a given bounded domain $\Omega \subset \mathbb{R}^n$ and a given Lipschitz function f, where ν denotes the outward unit normal vector about $\partial\Omega$ and $\langle \cdot, \cdot \rangle$ the scalar product in $\mathbb{R}^n$, then Ω must be a ball, [19]. In the Euclidean space, round balls are in fact not only extremal domains, but also minimizers for the first eigenvalue of the Laplacian with 0 Dirichlet boundary condition in the class of domains with the same volume. This follows from the Faber–Krähn inequality,

$$\lambda_\Omega \geq \lambda_{B^n(\Omega)} \tag{3}$$

where $B^n(\Omega)$ is a ball of $\mathbb{R}^n$ with the same volume as Ω, because equality holds in (3) if and only if $\Omega = B^n(\Omega)$, see [3] and [9].

Nevertheless, very few results are known about extremal domains in a Riemannian manifold. The result of J. Serrin, based on the moving plane argument introduced by A. D. Alexandrov in [1], uses strongly the symmetry of the Euclidean space, and naturally it fails in other geometries. The classification of extremal domains is then achieved in the Euclidean space, but it is completely open in a general Riemannian manifold.

For small volumes, a method to build new examples of extremal domains in some Riemannian manifolds has been developed in [12] by F. Pacard and P. Sicbaldi. They proved that when the Riemannian manifold has a nondegenerate critical point of the scalar curvature, then it is possible to build extremal domains of any given small enough volume, and such domains are close to geodesic balls centered at the nondegenerate critical point of the scalar curvature. The method fails if the Riemannian method does not have a nondegenerate critical point of the scalar curvature.

In this paper we improve the result of F. Pacard and P. Sicbaldi by eliminating the hypothesis of the existence of a nondegenerate critical point for the scalar curvature. In particular, we are able to build extremal domains of small volume in every compact Riemannian manifold.

For $\epsilon > 0$, we denote by $B^g_\epsilon(p) \subset M$ the geodesic ball of center $p \in M$ and radius ϵ. We denote by $B_\epsilon \subset \mathbb{R}^n$ the Euclidean ball of radius ϵ centered at the origin. The main result of the paper is the following:

Theorem 1.1. *Let M be a compact Riemannian manifold of dimension $n \geq 2$. There exist $\epsilon_0 > 0$ and a smooth function*

$$\Phi : M \times (0, \epsilon_0) \longrightarrow \mathbb{R}$$

such that:

(1) *For all $\epsilon \in (0, \epsilon_0)$, if p is a critical point of the function $\Phi(\cdot, \epsilon)$ then there exists an extremal domain $\Omega_\epsilon \subset M$, containing p, whose volume is equal to the Euclidean volume of B_ϵ. Moreover, there exists $c > 0$ and, for all $\epsilon \in (0, \epsilon_0)$, the boundary of Ω_ϵ is a normal graph over $\partial B^g_\epsilon(p)$ for some function $v(p, \epsilon)$ with*

$$\|v(p, \epsilon)\|_{\mathcal{C}^{2,\alpha}(\partial B^g_\epsilon(p))} \leq c\, \epsilon^3 .$$

(2) *There exists a function $\mathbf{r}$ defined on M that can be written as*

$$\mathbf{r} = K_1 \, \|Riem\|^2 + K_2 \, \|Ric\|^2 + K_3 \, R^2 + K_4 \, \Delta_g R$$

where Riem, Ric, R denote respectively the Riemann curvature tensor, the Ricci curvature tensor and the scalar curvature of (M, g), and K_1, K_2, K_3 and K_4 are constants depending only on n, such that for all $k \geq 0$

$$\|\Phi(p, \epsilon) - R_p - \epsilon^2 \, \mathbf{r}_p\|_{C^k(M)} \leq c_k \, \epsilon^3$$

for some constant $c_k > 0$ which does not depend on $\epsilon \in (0, \epsilon_0)$ (the subscript p means that we evaluate the function at p).

(3) *The following expansion holds:*

$$\begin{aligned} \lambda_{\Omega_\epsilon} &= \lambda_1 \, \epsilon^{-2} - \frac{n(n+2) + 2\lambda_1}{6n(n+2)} \, \Phi(p, \epsilon) \\ &= \lambda_1 \, \epsilon^{-2} - \frac{n(n+2) + 2\lambda_1}{6n(n+2)} \, \big(R_p + \epsilon^2 \, \mathbf{r}_p\big) + \mathcal{O}(\epsilon^3) \end{aligned}$$

where λ_1 is the first Dirichlet eigenvalue of the unit Euclidean ball.

The explicit computation of the constants K_i is given in section 7 (formulas (30)). We remark that if M is compact, then there exists always a critical point of $\Phi(\cdot, \epsilon)$, and then we have small extremal domains obtained as perturbation of small geodesic balls in every compact Riemannian manifold without boundary.

If M is not compact, the result holds on any relatively compact open set U for some $\epsilon_0 = \epsilon_0(U)$ and the function Φ is well defined on

$$\bigcup_{U \subset M} \big(U \times (0, \epsilon_0(U))\big) .$$

Let us explain briefly the construction of the function $\Phi(p,\epsilon)$. Firstly, we will show that for all point $p \in M$, and all ϵ small enough, there exists a function $v(p,\epsilon)$ defined on $\partial B^g_\epsilon(p)$ such that the domain $\Omega_{p,\epsilon}$ bounded by the normal graph of $v(p,\epsilon)$ over $\partial B^g_\epsilon(p)$ has the same volume of the Euclidean ball B_ϵ and the property that the Neuman data of the first eigenfunction of the Laplace-Beltrami operator over $\Omega_{p,\epsilon}$, seen up to a natural diffeomorphism as a function on the unit sphere, is the restriction of a linear function. Such domain $\Omega_{p,\epsilon}$ is then in some sense "close" to be extremal. Secondly, we will prove that $\Omega_{p,\epsilon}$ is extremal if and only if p is a critical point of the function $p \mapsto \lambda_1(\Omega_{p,\epsilon})$. The function $\Phi(.,\epsilon)$ is given, up to a constant, exactly by the function $p \mapsto \lambda_1(\Omega_{p,\epsilon})$.

It is clear that Theorem 1.1 generalizes the result in [12] because the construction of extremal domains does not require the existence of a nondegenerate critical point of the scalar curvature. In fact, if the scalar curvature function R has a nondegenerate critical point p_0, then for all ϵ small enough there exists a critical point $p = p(\epsilon)$ of $\Phi(\cdot,\epsilon)$ such that

$$\mathrm{dist}(p,p_0) \leq c\,\epsilon^2.$$

and then the geodesic ball $B^g_\epsilon(p)$ can be perturbed in order to obtain an extremal domain. We recover in this case the result in [12], but with a better estimation of the distance of p to p_0 (in [12] the distance between p and p_0 is bounded by $c\,\epsilon$). In particular, we have the *p-independent* expansion

$$\lambda_{\Omega_\epsilon} = \lambda_1\,\epsilon^{-2} - \frac{n(n+2)+2\lambda_1}{6n(n+2)}\,R_{p_0} + \mathcal{O}(\epsilon^2)$$

The result in [12] can not be applied to some natural metrics as an Einstein metric, i.e when $Ric = k\,g$ for some constant k, or simply a constant scalar curvature one. In the case where R is a constant function, one gets the existence of extremal domains close to any nondegenerate critical point of the function $\mathbf{r}$. In the particular case where the metric g is Einstein we obtain extremal domains close to any nondegenerate critical point of the function (we will see that $K_1 \neq 0$)

$$p \to \|Riem_p\|^2\,.$$

In order to put the result in perspective let us digress slightly. The solutions of the isoperimetric problem

$$I_\kappa := \min_{\Omega\subset M\,:\,\mathrm{Vol}_g\,\Omega=\kappa} \mathrm{Vol}_{g_{\mathrm{in}}}\,\partial\Omega$$

are (where they are smooth enough) constant mean curvature hypersurfaces (here g_{in} denotes the induced metric on the boundary of Ω). In fact, constant mean curvature are the critical points of the area functional

$$\Omega \to \mathrm{Vol}_{g_{\mathrm{in}}}\,\partial\Omega$$

under a volume constraint $\mathrm{Vol}_g\,\Omega = \kappa$. Now, it is well known (see [3], [9] and [10]) that the determination of the isoperimetric profile I_κ is related to the Faber-Krähn profile, where one looks for the least value of the first eigenvalue of the Laplace-Beltrami operator amongst domains with prescribed volume

$$FK_\kappa := \min_{\Omega\subset M \,:\, \mathrm{Vol}_g \Omega=\kappa} \lambda_\Omega$$

A smooth solution to this minimizing problem is an extremal domain, and in fact extremal domains are the critical points of the functional

$$\Omega \to \lambda_\Omega$$

under a volume constraint $\mathrm{Vol}_g \Omega = \kappa$.

The result by F. Pacard and P. Sicbaldi [12] had been inspired by some parallel results on the existence of constant mean curvature hypersurfaces in a Riemannian manifold M. In fact, R. Ye built in [22] constant mean curvature topological spheres which are close to geodesic spheres of small radius centered at a nondegenerate critical point of the scalar curvature, and the result of F. Pacard and P. Sicbaldi can be considered the parallel of the result of R. Ye in the context of extremal domains. The method used in [12] is based on the study of the operator that to a domain associates the Neumann value of its first eigenfunction, which is a nonlocal first order elliptic operator. This represents a big difference with respect to the result of R. Ye, where the operator to study was a local second order elliptic operator.

In a recent paper, [13], F. Pacard and X. Xu generalise the result of R. Ye by eliminating the hypothesis of the existence of a nondegenerate critical point of the scalar curvature function. For every ϵ small enough, they are able to build a small topological sphere of constant mean curvature equal to $\frac{n-1}{\epsilon}$ by perturbing a small geodesic ball centered at a critical point of a certain function defined on M which is close to the scalar curvature function. For this, they use the variational characterization of constant H_0 mean curvature hypersurfaces as critical points of the functional

$$S \to \mathrm{Vol}_{g_{\mathrm{in}}}(S) - H_0 \, \mathrm{Vol}_g(D_S)$$

in the class of topological sphere, where D_S is the domain enclosed by S, see [13].

Our construction is based on some ideas of [13]. For this, we use the variational characterization of extremal domains. The main difference and difficulties with respect to the result of F. Pacard and X. Xu lie in the fact that there does not exist an explicit formulation to compute the first eigenvalue of a domain while there exists an explicit formulation to compute the volume of a surface.

Our result shows once more the similarity between constant mean curvature hypersurfaces and extremal domains. The deep link between such two objects has been underlined also in [16] and [17].

It is important to remark that P. Sicbaldi was able to build extremal domains of big volume in some compact Riemannian manifold without boundary by perturbing the complement of a small geodesic ball centered at a nondegenerate critical point of the scalar curvature function, see [20]. As in the case of small volume domains, the existence of a nondegenerate critical point of the scalar curvature function is required (and such result requires also that the dimension of the manifold is at least 4). It would be interesting to adapt our result in order to build extremal domains of big volume in any compact Riemannian manifold without boundary by perturbing the complement of small geodesic balls of radius ϵ centered at a critical point of the function $\Phi(\cdot, \epsilon)$ or some other similar function. This result would allow for example to obtain extremal domains Ω_ϵ that are given by the complement of a small topological ball in a flat 2-dimensional torus, and by the characterization of extremal domains this would lead to a nontrivial solution of (2), with $f(t) = \lambda\, t$, in the universal covering $\tilde{\Omega}_\epsilon$ of Ω_ϵ, which is a nontrivial unbounded domain of $\mathbb{R}^2$. Up to our knowledge the existence of this unbounded domain is not known. Remark that $\tilde{\Omega}_\epsilon$ is a double periodic domain, made by the complement of a infinitely countable union of topological balls. The existence of $\tilde{\Omega}_\epsilon$ would establish once more the strong link between extremal domains and constant mean curvature surfaces, via the double periodic constant mean curvature surfaces (see [6], [15] and [14]).

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2. Notations and preliminaries

Let Ω_0 be a smooth bounded domain in M. We say that $\{\Omega_t\}_{t\in(-t_0,t_0)}$ is a deformation of Ω_0 if there exists a vector field Ξ such that $\Omega_t = \xi(t, \Omega_0)$ where $\xi(t, \cdot)$ is the flow associated to Ξ, namely

$$\frac{d\xi}{dt}(t,p) = \Xi(\xi(t,p)) \qquad \text{and} \qquad \xi(0,p) = p\,.$$

In this case we say that Ξ is the vector field that generates the deformation. The deformation is said to be volume preserving if the volume of Ω_t does not depend on t. If $\{\Omega_t\}_{t\in(-t_0,t_0)}$ is a deformation of Ω_0, and λ_{Ω_t} and u_t are respectively the first eigenvalue and the first eigenfunction (normalized to be positive and have $L^2(\Omega_t)$ norm equal to 1) of $-\Delta_g$ on Ω_t with zero Dirichlet boundary condition, both applications $t \longmapsto \lambda_{\Omega_t}$ and $t \longmapsto u_t$ inherit the regularity of the deformation of Ω_0. These facts are standard and follow at once from the implicit function theorem together with the fact that the least eigenvalue of the Laplace-Beltrami operator with 0 Dirichlet boundary condition is simple.

A domain Ω_0 is an *extremal domain* (for the first eigenvalue of $-\Delta_g$ with 0 Dirichlet boundary condition) if for any volume preserving deformation $\{\Omega_t\}_{t\in(-t_0,t_0)}$ of Ω_0, we have

$$\left.\frac{\mathrm{d}\lambda_{\Omega_t}}{\mathrm{d}t}\right|_{t=0} = 0\,.$$

Assume that $\{\Omega_t\}_t$ is a perturbation of a domain Ω_0 generated by the vector field Ξ. The outward unit normal vector field to $\partial\Omega_t$ is denoted by ν_t. We have the following result, whose proof can be found in [2] or in [12]:

Proposition 2.1. *(Garabedian – Schiffer, El Soufi – Ilias). The derivative of the first eigenvalue with respect to the deformation of the domain is given by*

$$\left.\frac{\mathrm{d}\lambda_{\Omega_t}}{\mathrm{d}t}\right|_{t=0} = -\int_{\partial\Omega_0} (g(\nabla u_0, \nu_0))^2 \;\; g(\Xi, \nu_0)\,\mathrm{dvol}_{g_{\rm in}}$$

This result allows to characterize extremal domains as the domains where there exists a positive solution to the overdetermined elliptic problem

$$\left\{\begin{array}{rcll} \Delta_g u + \lambda\, u & = & 0 & \text{in} \quad \Omega \\[2mm] u & = & 0 & \text{on} \quad \partial\Omega \\[2mm] g(\nabla u, \nu) & = & \text{constant} & \text{on} \quad \partial\Omega \end{array}\right. \tag{4}$$

for a positive constant λ, where ν is the outward unit normal vector about $\partial\Omega$. The proof of this fact follows directly from Proposition 2.1, but can be found also in [12].

Given a point $p \in M$ we denote by $E_1, \ldots, E_n$ an orthonormal basis of the tangent plane T_pM. Geodesic normal coordinates $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$ at p are defined by

$$X(x) := \mathrm{Exp}_p^g \left(\sum_{j=1}^n x^j\, E_j \right) \in M$$

where Exp_p^g is the exponential map at p for the metric g.

It will be convenient to identify $\mathbb{R}^n$ with T_pM and S^{n-1} with the unit sphere in T_pM. If $x := (x^1, \ldots, x^n) \in \mathbb{R}^n$, we set

$$\Theta(x) := \sum_{i=1}^n x^i\, E_i \in T_pM\,. \tag{5}$$

It corresponds to the vector of T_pM whose coordinates in the basis $(E_1, ..., E_n)$ are x. Given a continuous function $f : S^{n-1} \longmapsto (0, +\infty)$ whose L^∞-norm is sufficiently small we can define

$$B_f^g(p) := \left\{ \mathrm{Exp}_p^g(\Theta(x)) \qquad : \qquad x \in \mathbb{R}^n \qquad 0 < |x| < f\left(\frac{x}{|x|}\right) \right\} \cup \{p\}\,.$$

For notational convenience, given a continuous function $f : S^{n-1} \to (0, \infty)$, we set

$$B_f := \{ x \in \mathbb{R}^n \qquad : \qquad 0 < |x| < f(x/|x|) \} \cup \{0\}\,.$$

When we do not indicate the metric as a superscript, we understand that we are using the Euclidean one. Similarly, we denote by Vol_g the volume in the metric g, by dvol_g the volume element in the metric g to integrate over a domain, by $\mathrm{dvol}_{g_{\rm in}}$ the volume element in the induced metric $g_{\rm in}$ to integrate over the boundary of a domain. When we do not indicate anything we understand that we are considering the Euclidean volume, or the Euclidean measure, or the measure induced by the Euclidean one on boundaries.

Our aim is to show that, for all $\epsilon > 0$ small enough, we can find a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}_g\, B^g_{\epsilon(1+v)}(p) = \mathrm{Vol}\, B_\epsilon = \epsilon^n\, \mathrm{Vol}\, B_1 = \epsilon^n\, \frac{\omega_n}{n}$$

(where ω_n is the Euclidean volume of the unit sphere S^{n-1}) and the overdetermined problem

$$\left\{ \begin{array}{rclll} \Delta_g\, \phi + \lambda\, \phi & = & 0 & \text{in} & B^g_{\epsilon(1+v)}(p) \\[2mm] \phi & = & 0 & \text{on} & \partial B^g_{\epsilon(1+v)}(p) \\[2mm] g(\nabla \phi, \nu) & = & \text{constant} & \text{on} & \partial B^g_{\epsilon(1+v)}(p) \end{array} \right. \tag{6}$$

has a non trivial positive solution for some positive constant λ, where ν is the unit normal vector field about $\partial B^g_{\epsilon(1+v)}(p)$.

Clearly, this problem does not make sense when $\epsilon = 0$. In order to bypass this problem, we observe that, considering the dilated metric $\bar g := \epsilon^{-2}\, g$, the above problem is equivalent to finding a point $p \in M$ and a function $v : S^{n-1} \longrightarrow \mathbb{R}$ such that

$$\mathrm{Vol}_{\bar g}\, B^{\bar g}_{1+v}(p) = \mathrm{Vol}\, B_1$$

and for which the overdetermined problem

$$\left\{ \begin{array}{rclll} \Delta_{\bar g}\, \bar\phi + \bar\lambda\, \bar\phi & = & 0 & \text{in} & B^{\bar g}_{1+v}(p) \\[2mm] \bar\phi & = & 0 & \text{on} & \partial B^{\bar g}_{1+v}(p) \\[2mm] \bar g(\nabla^{\bar g} \bar\phi, \bar\nu) & = & \text{constant} & \text{on} & \partial B^{\bar g}_{1+v}(p) \end{array} \right. \tag{7}$$

has a non trivial positive solution for some positive constant $\bar\lambda$, where $\bar\nu$ is the unit normal vector field about $\partial B^{\bar g}_{1+v}(p)$. Taking in account that the functions ϕ and $\bar\phi$ have L^2-norm equal to 1, we have that the relation between the solutions of the two problems is simply given by

$$\phi = \epsilon^{-n/2}\, \bar\phi$$

and

$$\lambda = \epsilon^{-2}\, \bar\lambda\, .$$

3. Some expansions in normal geodesic coordinates

We specify that through this paper we consider the following definition of the Riemann curvature tensor:

$$Riem(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where ∇ denotes the Levi-Civita connection on the manifold M.

Geodesic normal coordinates are very useful because there exists a well known formula for the expansion of the coefficients of a metric near the center of such coordinates, see [21], [11] or [18]. At the point of coordinate x, the following expansion holds[1]:

$$
\begin{aligned}
g_{ij} &= \delta_{ij} - \frac{1}{3} R_{ikj\ell}\, x^k\, x^\ell - \frac{1}{6} R_{ikjl,m}\, x^k\, x^\ell\, x^m \\
&\quad - \frac{1}{20} R_{ikjl,m\sigma}\, x^k\, x^\ell\, x^m\, x^\sigma + \frac{2}{45} R_{ikj\ell}\, R_{imj\sigma}\, x^k\, x^\ell\, x^m\, x^\sigma + \mathcal{O}(|x|^5)
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
R_{ikj\ell} &= g\big(Riem_p(E_k, E_i)\, E_j, E_\ell\big) \\
R_{ikj\ell,m} &= g\big((\nabla_{E_m} Riem)_p(E_k, E_i)\, E_j, E_\ell\big) \\
R_{ikj\ell,m\sigma} &= g\big((\nabla_{E_\sigma} \nabla_{E_m} Riem)_p(E_k, E_i)\, E_j, E_\ell\big)\,,
\end{aligned}
$$

and the subscript p means that we evaluate the quantity at p. In (8) the Einstein notation is used (i.e., we do a summation on every index appearing up and down). Such notation will be always used through this paper.

This expansion allows to obtain other expansions, as those of the volume of a geodesic ball, or the first eigenvalue and the first eigenfunction on a geodesic ball. In order to recall such expansions, let us introduce some notations. Let us denote by λ_1 the first eigenvalue of the Laplacian in the unit ball B_1 with zero Dirichlet boundary condition. We denote by ϕ_1 the associated eigenfunction

$$
\left\{
\begin{array}{rcll}
\Delta \phi_1 + \lambda_1\, \phi_1 &=& 0 & \text{in} \quad B_1 \\[2mm]
\phi_1 &=& 0 & \text{on} \quad \partial B_1
\end{array}
\right.
\tag{9}
$$

normalized to be positive and have $L^2(B_1)$ norm equal to 1. It is clear that ϕ_1 is a radial function $\phi_1(x) = \phi_1(|x|)$. We denote $r = |x|$.

We recall now some expansions we will need later, whose proofs can be deduced from (8). We refer to [13] and [8] for the proofs. For the volume of a geodesic ball of radius ϵ we have:

$$
\epsilon^{-n}\, \mathrm{Vol}_g\, B_\epsilon^g(p) = \frac{\omega_n}{n} + W_0\, \epsilon^2 + W\, \epsilon^4 + \mathcal{O}(\epsilon^5),
\tag{10}
$$

where

$$
\begin{aligned}
W_0 &= -\frac{\omega_n}{6n\,(n+2)}\, R_p \\
W &= \frac{\omega_n}{360\, n\,(n+2)\,(n+4)}\, \big(-3\, \|Riem_p\|^2 + 8\, \|Ric_p\|^2 + 5\, R_p^2 - 18\, (\Delta_g R)_p\big)
\end{aligned}
\tag{11}
$$

[1]We choose the convention of [21], some sign in the development are different from those in [13] or [12] because of a different choice of the definition of R_{ijkl}

For the first eigenvalue of the Laplace-Beltrami operator with 0 Dirichlet boundary condition on a geodesic ball of radius ϵ we have:

$$\epsilon^2 \, \lambda_{B^g_\epsilon(p)} = \lambda_1 + \Lambda_0 \, \epsilon^2 \, + \Lambda \, \epsilon^4 + \mathcal{O}(\epsilon^5) \tag{12}$$

where

$$\begin{aligned} \Lambda_0 &= -\frac{R_p}{6} \\ \Lambda &= -\frac{c^2}{n(n+2)} \left(3 \, \|Riem_p\|^2 + \frac{35}{18} \, \|Ric_p\|^2 + \frac{5n-3}{18n} \, R_p^2 + \frac{1}{5} (\Delta_g R)_p \right) \end{aligned} \tag{13}$$

and the constant c^2 is given by

$$c^2 = -\int_0^1 \phi_1 \, \partial_r \phi_1 \, r^{n+2} \, \mathrm{d}r = \frac{n+2}{2} \int_0^1 \phi_1^2 \, r^{n+1} \, \mathrm{d}r$$

For the associate eigenfunction ϕ in the geodesic ball $B^g_\epsilon(p)$ normalized to be positive and with L^2-norm equal to 1, we have

$$\epsilon^{n/2} \, \phi(q) = \phi_1(y) + \left[\left(R_{ij} \, y^i \, y^j - \frac{R}{n} |y|^2 \right) \frac{\phi_1}{12} + R \; G_2(|y|) \right] \, \epsilon^2 + \mathcal{O}(\epsilon^3) \tag{14}$$

where q is the point of M whose geodesic coordinates are $\epsilon \, y$ for $y \in B_1$, and G_2 is defined implicitly as a solution of an ODE in [8]. Although we do not need its expression, for completeness we recall it: if we solve such ODE we found

$$G_2(r) = \frac{1}{12 \, n} \, r^2 \, \phi_1(r) - c^2 \, \frac{\omega_n}{6n \, (n+2)} \, \phi_1(r) \, . \tag{15}$$

4. Known results

Our aim is to perturbe the boundary of a small ball $B^{\bar g}_1(p)$ with a function v in order to obtained an extremal domain $B^{\bar g}_{1+v}(p)$. The natural space for the function v is $C^{2,\alpha}(S^{n-1})$ but not all functions in this space are admissible because v must satisfy also the condition

$$\mathrm{Vol}_{\bar g} \, B^{\bar g}_{1+v}(p) = \mathrm{Vol} \, B_1$$

In order to have a space of admissible functions not depending on the point p, we use a result proved in [12], that allows to use as space of admissible function the space

$$C^{2,\alpha}_m(S^{n-1}) = \left\{ \bar v \in C^{2,\alpha}(S^{n-1}) \qquad : \qquad \int_{S^{n-1}} \bar v = 0 \right\}$$

The result is the following:

Proposition 4.1. *(Pacard – Sicbaldi [12]) Let $p \in M$. For all ϵ small enough and all function $\bar v \in C^{2,\alpha}_m(S^{n-1})$ whose $C^{2,\alpha}$-norm is small enough there exist a unique positive function $\bar\phi = \bar\phi(p, \epsilon, \bar v) \in C^{2,\alpha}(B^{\bar g}_{1+v}(p))$, a constant $\bar\lambda = \bar\lambda(p, \epsilon, \bar v) \in \mathbb{R}$ and a constant $v_0 = v_0(p, \epsilon, \bar v) \in \mathbb{R}$ such that*

$$\mathrm{Vol}_{\bar g} \, B^{\bar g}_{1+v}(p) = \mathrm{Vol} \, B_1$$

where $v := v_0 + \bar v$ *and* $\bar\phi$ *is a solution to the problem*

$$
\left\{
\begin{array}{rcll}
\Delta_{\bar g}\, \bar\phi + \bar\lambda\, \bar\phi & = & 0 & \text{in} \quad B^{\bar g}_{1+v}(p) \\[3mm]
\bar\phi & = & 0 & \text{on} \quad \partial B^{\bar g}_{1+v}(p)
\end{array}
\right.
\tag{16}
$$

normalized by

$$
\int_{B^{\bar g}_{1+v}(p)} \bar\phi^2 \, \mathrm{dvol}_{\bar g} = 1.
$$

In addition $\bar\phi$, $\bar\lambda$ *and* v_0 *depend smoothly on the function* $\bar v$ *and the parameter* ϵ *and* $\bar\phi = \phi_1$, $\bar\lambda = \lambda_1$ *and* $v_0 = 0$ *when* $\epsilon = 0$ *and* $\bar v \equiv 0$. *Moreover* $v_0(p, \epsilon, 0) = \mathcal{O}(\epsilon^2)$.

Instead of working on a domain depending on the function $v = v_0 + \bar v$, it will be more convenient to work on a fixed domain B_1 endowed with a metric depending on both ϵ and the function v. This can be achieved by considering the parametrization of $B^{\bar g}_{1+v}(p)$ given by

$$
Y(y) := \mathrm{Exp}^{\bar g}_p \left(\left(1 + v_0 + \chi(y)\, \bar v \left(\frac{y}{|y|} \right) \right) \sum_i y^i \, E_i \right)
$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$. Hence the coordinates we consider from now on are $y \in B_1$ with the metric $\hat g := Y^* \bar g$.

Up to some multiplicative constant, the problem we want to solve can now be rewritten in the form

$$
\left\{
\begin{array}{rcll}
\Delta_{\hat g}\, \hat\phi + \hat\lambda\, \hat\phi & = & 0 & \text{in} \quad B_1 \\[3mm]
\hat\phi & = & 0 & \text{on} \quad \partial B_1
\end{array}
\right.
\tag{17}
$$

with

$$
\int_{B_1} \hat\phi^2 \, \mathrm{dvol}_{\hat g} = 1
\tag{18}
$$

and

$$
\mathrm{Vol}_{\hat g}(B_1) = \mathrm{Vol}\, B_1
\tag{19}
$$

When $\epsilon = 0$ and $\bar v \equiv 0$, a solution of (17) is given by $\hat\phi = \phi_1$, $\hat\lambda = \lambda_1$ and $v_0 = 0$. In the general case, the relation between the function $\bar\phi$ and the function $\hat\phi$ is simply given by

$$
Y^* \bar\phi = \hat\phi \qquad \text{and} \qquad \bar\lambda = \hat\lambda\,.
$$

We define the operator

$$
F(p, \epsilon, \bar v) = \hat g(\hat\nabla \hat\phi, \hat\nu)\Big|_{\partial B_1} - \frac{1}{\omega_n} \int_{\partial B_1} \hat g(\hat\nabla \hat\phi, \hat\nu)
$$

where $\hat\nu$ is the the unit normal vector field to ∂B_1 using the metric $\hat g$ and $(\bar\phi, v_0)$ is the solution of (16) provided by the Proposition 4.1. Recall that $v = v_0 + \bar v$. Schauder's estimates imply that F is well defined from a neighbourhood of $M \times \{0\} \times \{0\}$ in $M \times [0,\infty) \times C^{2,\alpha}_m(S^{n-1})$ into $C^{1,\alpha}_m(S^{n-1})$ (the space $C^{1,\alpha}_m(S^{n-1})$ is naturally the space of functions in $C^{1,\alpha}(S^{n-1})$ whose mean is 0). Our aim is to find $(p,\epsilon,\bar v)$ such that $F(p,\epsilon,\bar v)=0$. Observe that, with this condition, $\bar\phi$ will be the solution to problem (7).

We also have the alternative expression for F, after canonical identification of $\partial B^{\bar g}_{1+v}(p)$ with S^{n-1},

$$F(p,\epsilon,\bar v) = \bar g(\bar\nabla\bar\phi,\bar\nu)\,|_{\partial B^{\bar g}_{1+v}} - \frac{1}{\omega_n}\int_{\partial B^{\bar g}_{1+v}} \bar g(\bar\nabla\bar\phi,\bar\nu)$$

where this time $\bar\nu$ denotes the unit normal vector field to $\partial B^{\bar g}_{1+v}$.

For all $\bar v \in \mathcal{C}^{2,\alpha}_m(S^{n-1})$ let ψ be the (unique) solution of

$$\left\{\begin{array}{rcll} \Delta\psi + \lambda_1\,\psi &=& 0 & \text{in} \quad B_1 \\ \psi &=& -c_1\,\bar v & \text{on} \quad \partial B_1 \end{array}\right. \tag{20}$$

which is $L^2(B_1)$-orthogonal to ϕ_1, where $c_1 := \partial_r\phi_1|_{r=1}$. Define

$$H(\bar v) := (\partial_r\psi + c_2\,\bar v)\,|_{\partial B_1} \tag{21}$$

where $c_2 = \partial^2_r\phi_1|_{r=1}$. We recall that the eigenvalues of the operator $-\Delta_{S^{n-1}}$ are given by $\mu_j = j\,(n-2+j)$ for $j\in\mathbb{N}$, and we denote by V_j the eigenspace associated to μ_j.

The following result shows that H is the linearization of F with respect to $\bar v$ at $\epsilon=0$ and $\bar v = 0$:

Proposition 4.2. *(Pacard – Sicbaldi, [12]) The operator obtained by linearizing F with respect to $\bar v$ at $\epsilon = 0$ and $\bar v = 0$ is*

$$H : C^{2,\alpha}_m(S^{n-1}) \longrightarrow C^{1,\alpha}_m(S^{n-1})$$

It is a self adjoint, first order elliptic operator. The kernel of H is given by V_1. Moreover there exists $c>0$ such that

$$\|w\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\|H(w)\|_{\mathcal{C}^{1,\alpha}(S^{n-1})}\,,$$

provided w is $L^2(S^{n-1})$-orthogonal to $V_0 \oplus V_1$.

Using the previous proposition and the fact that V_1 is the restriction on the sphere of affine functions, the implicit function theorem gives directly the following:

Proposition 4.3. *(Pacard – Sicbaldi, [12]) There exists $\epsilon_0>0$ such that, for all $\epsilon\in[0,\epsilon_0]$ and for all $p\in M$, there exists a unique function $\bar v = \bar v(p,\epsilon) \in C^{2,\alpha}_m(S^{n-1})$, orthogonal to $V_0\oplus V_1$, and a vector $a = a(p,\epsilon)\in\mathbb{R}^n$ such that*

$$F(p,\epsilon,\bar v) + \langle a,\cdot\rangle = 0 \tag{22}$$

The function $\bar v$ and the vector a depend smoothly on p and ϵ and we have

$$|a| + \|\bar v\|_{\mathcal{C}^{2,\alpha}(S^{n-1})} \le c\,\epsilon^2$$

In other word, for every point $p \in M$ it is possible to perturbe the small ball $B_1^{\bar{g}}(p)$ in a domain $B_{1+v}^{\bar{g}}(p)$, whose volume did not change, but with the (strong) property that $F(p,\epsilon,\bar{v})$ (i.e. the Neumann data of its first eigenfunction minus its mean) is the restriction of a linear function $\langle a,\cdot\rangle$ on S^{n-1}. It is important to underline that this result does not depend on the geometry of the manifold, because it is true for every point p.

Now, we have to find the good point p for which such linear function $\langle a,\cdot\rangle$ is the 0 function. And in this research we will see the geometry of the manifold.

5. Construction of small extremal domains

For $p \in M$, let us define the function

$$\Psi_\epsilon(p) := \hat{\lambda} = \hat{\lambda}(p,\epsilon,\bar{v}(p,\epsilon))$$

where $\hat{\lambda}$ is given by (17) taking $\bar{v} = \bar{v}(p,\epsilon)$ given by Proposition 4.3.

Proposition 5.1. *For ϵ small enough, the domain $B_{1+v(p,\epsilon)}^{\bar{g}}(p)$ is extremal if and only if p is a critical point of Ψ_ϵ, where $v(p,\epsilon) = v_0(p,\epsilon,\bar{v}(p,\epsilon)) + \bar{v}(p,\epsilon)$.*

Proof. Recall that by definition

$$F(p,\epsilon,\bar{v}(p,\epsilon)) = \hat{g}(\hat{\nu},\hat{\nabla}\hat{\phi}) - b$$

where

$$b = b(p,\epsilon) := \frac{1}{\mathrm{Vol}_{\hat{g}_{\mathrm{in}}}(\partial B_1)} \int_{\partial B_1} \hat{g}(\hat{\nu},\hat{\nabla}\hat{\phi})\, \mathrm{dvol}_{\hat{g}_{\mathrm{in}}}$$

and

$$\int_{\partial B_1} F \,\mathrm{dvol}_{\hat{g}_{\mathrm{in}}} = 0\,.$$

Moreover we know that

$$F(p,\epsilon,\bar{v}(p,\epsilon)) + \langle a(p,\epsilon),\cdot\rangle = 0.$$

In particular the domain $B_{1+v(p,\epsilon)}^{\bar{g}}(p)$ is extremal if and only if $a(p,\epsilon) = 0$.

Let us now compute the differential of Ψ_ϵ. Let $\Xi \in T_pM$ and

$$q := \mathrm{Exp}_p(t\Xi).$$

For t small enough, the boundary of $B_{1+v(q,\epsilon)}^{\bar{g}}(q)$ can be written as a normal graph over the boundary of $B_{1+v(p,\epsilon)}^{\bar{g}}(p)$ for some function f, depending on p,ϵ,t and Ξ, and smooth on t. This defines a vector field on $\partial B_{1+v(p,\epsilon)}^{\bar{g}}(p)$ by

$$Z := \left.\frac{\partial f}{\partial t}\right|_{t=0} \bar{\nu}$$

where $\bar{\nu}$ is the normal of $\partial B_{1+v(p,\epsilon)}^{\bar{g}}(p)$. Let X be the vector field obtained by parallel transport of Ξ from geodesic issued from p. As the metric $\bar{g}$ is close to the Euclidean one for ϵ small, there exists a constant c such that for all ϵ small enough and any Ξ the estimation

$$\|Z - X\|_{\bar{g}} \leq c\|\Xi\|_{\bar{g}}.$$

holds. The variation of the first eigenvalue, see Proposition 2.1, gives

$$D_p\Psi_\epsilon(\Xi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \Psi_\epsilon(q) = -\int_{\partial B_1} [\hat{g}(\hat{\nabla}\hat{\phi}, \hat{\nu})]^2 \, \hat{g}(\hat{Z}, \hat{\nu}) \, \mathrm{dvol}_{\hat{g}_{\rm in}}.$$

We thus obtain

$$D_p\Psi_\epsilon(\Xi) = -\int_{\partial B_1} [-\langle a(p,\epsilon), \cdot\rangle + b]^2 \, \hat{g}(\hat{Z}, \hat{\nu}) \, \mathrm{dvol}_{\hat{g}_{\rm in}} \tag{23}$$

Recall that the variation we made is volume preserving, i.e.

$$\int_{\partial B_1} \hat{g}(\hat{Z}, \hat{\nu}) \, \mathrm{dvol}_{\hat{g}_{\rm in}} = 0\,.$$

Then it is easy to see that if $a = 0$ then $D_p\Psi_\epsilon = 0$. This proves one implication.

For the reverse implication, assume now that $D_p\Psi_\epsilon = 0$. From (23) we have

$$2b \int_{\partial B_1} \langle a(p,\epsilon), \cdot\rangle \, \hat{g}(\hat{Z}, \hat{\nu}) \, \mathrm{dvol}_{\hat{g}_{\rm in}} = \int_{\partial B_1} \langle a(p,\epsilon), \cdot\rangle^2 \, \hat{g}(\hat{Z}, \hat{\nu}) \, \mathrm{dvol}_{\hat{g}_{\rm in}} \tag{24}$$

for all Ξ. It is easy to see that for all ϵ small enough there exists a constant c such that

$$|\hat{g}(\hat{Z}, \hat{\nu}) - \langle \Xi, \cdot\rangle| \leq c\,\epsilon\,\|\Xi\|_g$$

(in fact the left hand side vanishes when $\epsilon = 0$, the metric $\hat{g}$ and the Euclidean one differ by terms of order ϵ^2 and the normal vectors differ by terms of order ϵ). Now we choose $\Xi = b\,a = b(p,\epsilon)\,a(p,\epsilon)$ and we get

$$\hat{g}(\hat{Z}, \hat{\nu}) = b\,\langle a, \cdot\rangle + \epsilon\,A$$

where $|A| \leq c\,\|ba\|_g$. Using this equality in equation (24), we deduce that for all ϵ small enought there exists a constant C independent on ϵ and a such that

$$2b^2 \int_{\partial B_1} \langle a, \cdot\rangle^2 \, \mathrm{dvol}_{\hat{g}_{\rm in}} \leq C\,|b|\,(\epsilon\,\|a\|^3 + \|a\|^3 + \epsilon\|a\|^2)\,.$$

Now the left hand side is bounded by below by $b^2\,\|a\|^2$, so finally we obtain

$$b^2\,\|a\|^2 \leq C\,|b|\,(\epsilon\,\|a\| + \|a\| + \epsilon)\,\|a\|^2\,.$$

Observe that $|b|$ is bounded away from zero by a uniform constant because when $\epsilon = 0$, $b \neq 0$. As $\|a\| = \mathcal{O}(\epsilon^2)$, then for ϵ small (recall $b \neq 0$) we obtain that $a = 0$ and this concludes the proof of the proposition. □

We now define

$$\Phi(p,\epsilon) = -\frac{6\,n\,(n+2)}{n\,(n+2) + 2\lambda_1}\,\frac{\Psi_\epsilon(p) - \lambda_1}{\epsilon^2}\,,$$

where λ_1 is the first eigenvalue of the euclidean unit ball. Propositions 4.3 and 5.1 completes the proof of the first part of Theorem 1.1. In the following sections, we will prove the second and the third parts of Theorem 1.1, and for this we have to find an expansion in power of ϵ for $\Psi_\epsilon(p)$. Such expansion will involve the geometry of the manifold.

6. Expansion of the first eigenvalue on perturbations of small geodesic balls

In this section we want to find an expansion of the first eigenvalue $\hat\lambda = \hat\lambda(p,\epsilon,\bar v)$ in power of ϵ and $\bar v$, where p is fixed in M. In a second time, we will use the function $\bar v = \bar v(p,\epsilon)$ given by Proposition 4.3 in order to find an expansion of $\hat\lambda(p,\epsilon,\bar v(p,\epsilon))$ in power of ϵ. Keeping in mind that we will have $\bar v = \mathcal{O}(\epsilon^2)$ we write formally

$$\begin{aligned}\hat\lambda(p,\epsilon,\bar v) &= \hat\lambda(p,0,0) + \partial_\epsilon\hat\lambda(p,0,0)\,\epsilon \\ &\quad + \partial_{\bar v}\hat\lambda(p,0,0)\,\bar v + \frac{1}{2}\,\partial_\epsilon^2\hat\lambda(p,0,0)\,\epsilon^2 \\ &\quad + \partial_\epsilon\partial_{\bar v}\hat\lambda(p,0,0)\,\epsilon\,\bar v + \frac{1}{6}\,\partial_\epsilon^3\hat\lambda(p,0,0)\,\epsilon^3 \\ &\quad + \frac{1}{2}\,\partial_{\bar v}^2\hat\lambda(p,0,0)\,\bar v^2 + \frac{1}{2}\,\partial_\epsilon^2\partial_{\bar v}\hat\lambda(p,0,0)\,\epsilon^2\,\bar v + \frac{1}{24}\,\partial_\epsilon^4\hat\lambda(p,0,0)\,\epsilon^4 \\ &\quad + \mathcal{O}(\epsilon^5)\end{aligned}$$

We thus study all of theses terms.

Lemma 6.1. *We have*

$$\begin{aligned}\partial_\epsilon\hat\lambda(p,0,0) &= 0 \\ \frac{1}{2}\,\partial_\epsilon^2\hat\lambda(p,0,0) &= -\frac{R_p}{6}\left(1 + 2\,\frac{\lambda_1}{n\,(n+2)}\right) =: \hat\Lambda_0 \\ \partial_\epsilon^3\hat\lambda(p,0,0) &= 0 \\ \frac{1}{24}\,\partial_\epsilon^4\hat\lambda(p,0,0) &= \Lambda + \lambda_1\left(\frac{2\,W}{\omega_n} - \frac{R_p^2}{36\,n^2\,(n+2)}\right) =: \hat\Lambda\end{aligned}$$

where the constants Λ and W are given in (11) and (13).

Proof. It suffices to find the expansion of $\hat\lambda(p,\epsilon,0)$ in power of ϵ. First we have to expand $v_0(p,\epsilon,0)$ and this can be done by using expansion (10), keeping in mind the definition of the metric $\hat g$ and the fact that when $\bar v = 0$ the constant v_0 is given by the relation

$$\mathrm{Vol}_{\hat g}\,B_1 = \mathrm{Vol}_{\bar g}\,B^{\bar g}_{1+v_0} = \epsilon^{-n}\mathrm{Vol}_g\,B_{\epsilon(1+v_0)} = \mathrm{Vol}\,B_1 = \frac{\omega_n}{n}$$

Using expansion (10) with ϵ replaced by $\epsilon(1+v_0)$, we find

$$v_0 = A_0\,\epsilon^2 + A\,\epsilon^4 + \mathcal{O}(\epsilon^5)$$

where

$$\begin{aligned} A_0 &= -\frac{W_0}{\omega_n} \\ A &= -\frac{1}{\omega_n}\left((n+2)\,A_0\,W_0 + \frac{n-1}{2}\,A_0^2\,\omega_n + W\right) \end{aligned}$$

Now we use expansion (12) replacing ϵ by $\epsilon(1+v_0)$. We obtain

$$\hat{\lambda} = \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 + \mathcal{O}(\epsilon^5)$$

where

$$\begin{aligned} \hat{\Lambda}_0 &= \Lambda_0 - 2\,\lambda_1\,A_0 = -\frac{R_p}{6}\left(1 + 2\,\frac{\lambda_1}{n\,(n+2)}\right) \\ \hat{\Lambda} &= \Lambda - \lambda_1\,(2A - 3\,A_0^2) = \Lambda + \lambda_1\left(\frac{2\,W}{\omega_n} - \frac{R_p^2}{36\,n^2\,(n+2)}\right) \end{aligned}$$

This concludes the proof of the result. □

Lemma 6.2. *We have*

$$\begin{aligned} \partial_{\bar{v}}\hat{\lambda}(p,0,0) &= 0 \\ \partial_{\bar{v}}^2\hat{\lambda}(p,0,0)(\bar{v},\bar{v}) &= -2\,c_1\int_{S^{n-1}}\bar{v}\,H(\bar{v}) \end{aligned}$$

where H is the operator of Proposition 4.2, whose expression is given by (21), and $c_1 := \partial_r\phi_1|_{r=1}$ is the constant defined in (20).

Proof. Let $\Omega_0 = B_1$ be the unit ball of $\mathbb{R}^n$, and let $\Omega_t = B_{(1+v_0+t\bar{v})}$, where we recall that $\int_{S^{n-1}}\bar{v} = 0$ and $v_0 = v_0(t)$ is chosen so that $\operatorname{Vol}\Omega_t = \operatorname{Vol}\Omega_0 = \frac{\omega_n}{n}$. We have

$$\partial_{\bar{v}}\hat{\lambda}(p,0,0)(\bar{v}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\lambda_{\Omega_t}$$

and

$$\partial_{\bar{v}}^2\hat{\lambda}(p,0,0)(\bar{v},\bar{v}) = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\lambda_{\Omega_t}$$

where λ_{Ω_t} is the first Dirichlet eigenvalue of Ω_t. The expansion of $\operatorname{Vol}\Omega_t$ directly proves that $v_0 = \mathcal{O}(t^2)$. In fact, in polar coordinates, we have

$$\begin{aligned} \operatorname{Vol}\Omega_t &= \int_{S^{n-1}}\int_0^{1+v_0(t)+t\,\bar{v}} r^{n-1}\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \frac{1}{n}\int_{S^{n-1}}(1+v_0(t)+t\,\bar{v})^n\,\mathrm{d}\theta \\ &= \frac{1}{n}\int_{S^{n-1}}\left[(1+v_0(t))^n + n\,(1+v_0)^{n-1}\,t\,\bar{v} + \mathcal{O}(t^2)\right]\,\mathrm{d}\theta \\ &= \frac{\omega_n\,(1+v_0(t))^n}{n} + \mathcal{O}(t^2) \end{aligned}$$

Differentiating this expression with respect to t, and keeping in mind that $v_0(0) = 0$, we obtain that $v_0(t) = \mathcal{O}(t^2)$. For $y \in \Omega_0$ and t small, let

$$h(t, y) = \left(1 + v_0 + t\, \chi(y)\, \bar{v}\left(\frac{y}{|y|}\right)\right) y$$

where χ is a cutoff function identically equal to 0 when $|y| \leq 1/2$ and identically equal to 1 when $|y| \geq 3/4$, so that $h(t, \Omega_0) = \Omega_t$. We will denote the t-derivative with a dot. Let $V(t, h(t, y)) = \dot{h}(t, y)$ be the first variation of the domain Ω_t. Let ν be the unit normal to $\partial\Omega_t$ and let $\sigma = \langle V, \nu\rangle$ the normal variation about $\partial\Omega_t$. Let λ be the first eigenvalue and ϕ the first eigenfunction of the Dirichlet Laplacian over Ω_t normalized in order to have L^2 norm equal to 1. From Proposition 2.1 we have

$$\dot{\lambda} = -\int_{\partial\Omega_t} (\partial_\nu \phi)^2\, \sigma$$

where $\partial_\nu \phi = \langle \nabla\phi, \nu\rangle$. At $t = 0$ and on the boundary, we have $\phi = \phi_1$, $\partial_\nu\phi = \partial_r \phi_1 = c_1$, $\sigma = \bar{v}$. Then $\dot{\lambda}(0) = 0$. This proves the first part of the Lemma.

We can use now equality (34) of Proposition 10.1 of the Appendix (with $f = (\partial_\nu\phi)^2\, \sigma$) in order to derivate this formula with respect to t. We obtain

$$\ddot{\lambda} = -\int_{\partial\Omega_t} \left[(\partial_\nu\phi)^2\, (\dot{\sigma} + \sigma\, \partial_\nu \sigma\, + \tilde{H}\, \sigma^2) + 2\sigma\, (\partial_\nu\phi\, \partial_\nu\dot{\phi} + \sigma\, \partial_\nu\phi\, \partial^2_\nu \phi)\right]$$

where $\tilde{H}$ is the mean curvature of $\partial\Omega_t$. Now the second variation of the volume of Ω_t is

$$\ddot{\mathrm{Vol}}\, \Omega_t = \int_{\partial\Omega_t} (\dot{\sigma} + \sigma\, \partial_\nu\sigma + \tilde{H}\sigma^2) = 0.$$

Such equation can be obtained differentiating equality (33) of Proposition 10.1 with $f = 1$, using equality (34) of Proposition 10.1 with $f = \sigma$. On the other hand, at $t = 0$ and on the boundary, we have $\phi = \phi_1$, $\partial_\nu\phi = \partial_r\phi_1 = c_1$, $\partial^2_\nu\phi = \partial^2_r\phi_1 = c_2 = -(n-1)c_1$, $\sigma = \bar{v}$. We claim that at $t = 0$ we have also $\dot{\phi} = \psi$, where ψ solve (20) and is $L^2(B_1)$-orthogonal to ϕ_1. This last claim can be easily proved by writing

$$\phi = \phi(t) = \phi_1 + t\, \psi + \mathcal{O}(t^2)\, .$$

Since $\lambda_{\Omega_t} = \lambda_1 + \mathcal{O}(t^2)$, differentiation of

$$\left\{\begin{array}{rcll} \Delta\, \phi(t) + \lambda_{\Omega_t}\, \phi(t) & = & 0 & \text{in} \quad \Omega_t \\ \phi(t) & = & 0 & \text{on} \quad \partial\Omega_t \end{array}\right.$$

with respect to t at $t = 0$ gives exactly (20). Moreover differentiation of

$$\int_{\Omega_t} \phi(t)^2 = 1$$

with respect to t at $t = 0$ implies that ψ is $L^2(B_1)$-orthogonal to ϕ_1. Our claim is then proved, and in conclusion we obtain

$$\ddot{\lambda}(0) = -2\, c_1 \int_{S^{n-1}} \bar{v}\,(\partial_r \psi + c_2 \bar{v}) = -2\, c_1 \int_{S^{n-1}} \bar{v}\, H(\bar{v}).$$

The proof of the Lemma follows at once. □

Lemma 6.3. *We have*

$$\begin{aligned} \partial_\epsilon \partial_{\bar{v}} \hat{\lambda}(p,0,0) &= 0 \\ \partial^2_\epsilon \partial_{\bar{v}} \hat{\lambda}(p,0,0)\, \bar{v} &= -\frac{c_1^2}{3} \int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\, \bar{v} \end{aligned}$$

where Θ *has been defined in (5),* $c_1 := \partial_r \phi_1|_{r=1}$ *is the constant defined in (20), and*

$$\mathring{Ric} = Ric - \frac{R}{n}\, g$$

is the traceless Ricci curvature.

In order to prove this lemma, we start with a preliminary result. The formulas for the geometric quantities we will consider are potentially complicated, and to keep notations short, we agree on the following: any expression of the form $L_p(v)$ denotes a linear combination of the function v together with its derivatives up to order 1, whose coefficients can depend on ϵ and there exists a positive constant c independent on $\epsilon \in (0,1)$ and on p such that

$$\|L_p(v)\|_{C^{1,\alpha}(S^{n-1})} \leq c\, \|v\|_{C^{2,\alpha}(S^{n-1})}\,;$$

similarly, given $a \in \mathbb{N}$, any expression of the form $Q_p^{(a)}(v)$ denotes a nonlinear operator in the function v together with its derivatives up to order 1, whose coefficients can depend on ϵ and there exists a positive constant c independent on $\epsilon \in (0,1)$ and on p such that

$$\|Q_p^{(a)}(v_1) - Q_p^{(a)}(v_2)\|_{C^{1,\alpha}(S^{n-1})} \leq c\, \left(\|v_1\|_{C^{2,\alpha}(S^{n-1})} + \|v_2\|_{C^{2,\alpha}(S^{n-1})}\right)^{a-1} \|v_2 - v_1\|_{C^{2,\alpha}(S^{n-1})}$$

provided $\|v_i\|_{C^{2,\alpha}(S^{n-1})} \leq 1$, for $i = 1, 2$.

Lemma 6.4. *We have*

$$\partial_{\bar{v}} v_0(p,\epsilon,0)(\bar{v}) = \frac{\epsilon^2}{6\,\omega_n} \int_{S^{n-1}} \mathring{Ric}(\Theta,\Theta)\, \bar{v} + \int_{S^{n-1}} [\mathcal{O}(\epsilon^5) + \epsilon^3\, L_p(\bar{v})]$$

where Θ *has been defined in (5).*

Proof. The expansion in ϵ and v for the volume of the perturbed geodesic ball $B^g_{\epsilon(1+v)}(p)$ is given in the Appendix of [13] (the corresponding notations with respect to [13] are

$B^g_{\epsilon\,(1+v)}(p) = B_{p,\epsilon}(-v)$ and $n = m+1$). We have:

$$
\begin{aligned}
\epsilon^{-n}\,\mathrm{Vol}(B^g_{\epsilon(1+v)}(p)) \;&=\; \frac{\omega_n}{n} + W_0\,\epsilon^2 + W\,\epsilon^4 \\
&\quad - \int_{S^{n-1}} v + \frac{n-1}{2}\int_{S^{n-1}} v^2 - \frac{1}{6}\,\epsilon^2 \int_{S^{n-1}} Ric_p(\Theta,\Theta)\, v \\
&\quad + \int_{S^{n-1}} \big(\mathcal{O}(\epsilon^5) + \epsilon^3\, L_p(v) + \epsilon^2\, Q^{(2)}_p(v) + Q^{(3)}_p(v)\big)
\end{aligned} \tag{25}
$$

where Θ has been defined in (5), and W_0, W are given by (11). Putting $v = v_0 + \bar v$ in expansion (25), where $\int_{S^{n-1}} \bar v = 0$ and v_0 is chosen in order that the volume of $B^g_{\epsilon\,(1+v)}(p)$ is equal to the volume of B_ϵ, we obtain

$$
\begin{aligned}
\epsilon^{-n}\,\mathrm{Vol}(B^g_{\epsilon(1+v)}(p)) \;&=\; \frac{\omega_n}{n} + W_0\,\epsilon^2 + W\,\epsilon^4 \\
&\quad + v_0\left[\omega_n\left(1 + \frac{n-1}{2}\,v_0\right) - \frac{1}{6}\,\epsilon^2 \int_{S^{n-1}} Ric_p(\Theta,\Theta)\right] \\
&\quad + \frac{n-1}{2}\int_{S^{n-1}} \bar v^2 - \frac{1}{6}\,\epsilon^2 \int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar v \\
&\quad + \int_{S^{n-1}} \big(\mathcal{O}(\epsilon^5) + \epsilon^3\, L_p(v) + \epsilon^2\, Q^{(2)}_p(v) + Q^{(3)}_p(v)\big)\,.
\end{aligned}
$$

In order to compute the expansion of

$$
\dot v_0 := \partial_{\bar v} v_0(p,\epsilon,0)(\bar v) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} v_0(p,\epsilon,s\bar v).
$$

we derivate with respect to s, at $s=0$, equality

$$
\mathrm{Vol}_g\, B^g_{\epsilon(1+v_0(p,\epsilon,s\bar v)+s\bar v)}(p) = \mathrm{Vol}\, B_\epsilon
$$

using the expansion above. Recall that we know $v_0(p,\epsilon,0) = \mathcal{O}(\epsilon^2)$. We find

$$
(1+\mathcal{O}(\epsilon^2))\,\omega_n\,\dot v_0 = \frac{1}{6}\,\epsilon^2 \int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar v + \int_{S^{n-1}} \big(\mathcal{O}(\epsilon^5) + \epsilon^3\, L_p(\bar v)\big)
$$

Finally

$$
\dot v_0 = \frac{1}{6\,\omega_n}\,\epsilon^2 \int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar v + \int_{S^{n-1}} \big(\mathcal{O}(\epsilon^5) + \epsilon^3\, L_p(\bar v)\big)
$$

This completes the proof of the Lemma. □

We are now able to prove Lemma 6.3.

Proof. (Lemma 6.3). We make a development up to power 2 in ϵ, of the function

$$
\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \hat\lambda(p,\epsilon,s\bar v)\,.
$$

From Proposition 2.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} \hat{\lambda}(p,\epsilon,s\bar{v}) = -\int_{\partial B_1} \hat{g}(V,\hat{\nu})(\hat{g}(\hat{\nabla}\hat{\phi},\hat{\nu}))^2 \mathrm{dvol}_{\hat{g}_{\mathrm{in}}}$$

where the deformation in a neighborhood of ∂B_1 is given by

$$h(s,y) = (1 + v_0(\epsilon, s\,\bar{v}) + s\,\bar{v})\,y$$

and

$$V(y) = \frac{\partial h}{\partial s}\bigg|_{s=0} = \left[\partial_{\bar{v}} v_0(p,\epsilon,0)(\bar{v}) + \bar{v}\left(\frac{y}{|y|}\right)\right] y.$$

In that formula, the term $\hat{g}(\hat{\nabla}\hat{\phi},\hat{\nu})$ is computed with $s = 0$ or equivalently $\bar{v} = 0$. From the definition of $\hat{g}$ and the expansion of the metric g, when $\bar{v} = 0$ we have

$$\begin{aligned} \hat{g}_{ij} &= (1 + v_0(\epsilon,0))^2 \left(\delta_{ij} - \frac{1}{3}\epsilon^2 R_{ikjl}\, y^k\, y^l + \mathcal{O}(\epsilon^3)\right) \\ \hat{\nu} &= (1 + v_0(\epsilon,0))^{-1}\,\partial_r = (1 + v_0(\epsilon,0))^{-1}\,\frac{y}{|y|} \end{aligned}$$

The expansion of $\hat{\phi}(p,\epsilon,0)$ is almost known: it suffices to replace ϵ by $\epsilon(1+v_0)$ in formula (14). We have

$$\hat{\phi} = \phi_1 + \epsilon^2 f_2 + \mathcal{O}(\epsilon^3)$$

where

$$f_2(y) = \left[R_{ij}\, y^i\, y^j - \frac{R_p}{n}|y|^2\right]\frac{\phi_1}{12} + R_p\, G_2(|y|).$$

Using the notation $R^j{}_k{}^m{}_l = g^{ja} g^{mb} R_{akbl}$ we thus have on ∂B_1

$$\begin{aligned} \hat{g}(\hat{\nabla}\hat{\phi},\hat{\nu}) &= (1 + v_0(\epsilon,0))^{-1}\left(\delta_{ij} - \frac{1}{3}\epsilon^2 R_{ikjl}\, y^k\, y^l\right)\cdot \\ &\quad \cdot y^i \left(\delta^{jp} + \frac{1}{3}\epsilon^2 R^j{}_k{}^m{}_l\, y^k\, y^l\right)\left(\frac{\partial}{\partial y^m}\phi_1 + \epsilon^2\frac{\partial}{\partial y^m} f_2\right) + \mathcal{O}(\epsilon^3) \\ &= (1 - v_0(\epsilon,0))^{-1}\, c_1 + \epsilon^2\,\partial_r f_2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

where on the boundary

$$\partial_r f_2(y) = \frac{c_1}{12}\left[R_{ij}\, y^i\, y^j - \frac{R_p}{n}\right] + R_p\, G_2'(1)\,.$$

Now we have to expand the measure on the boundary. This is classical and can be done directly from expansion (8). We have

$$\mathrm{dvol}_{\hat{g}_{\mathrm{in}}}|_{\partial B_1} = (1+v_0)^n\left[1 - \frac{1}{6}\, Ric_p(\Theta,\Theta)\,\epsilon^2 + \mathcal{O}(\epsilon^2)\right]\mathrm{dvol}|_{S^{n-1}}$$

where Θ has been defined in (5) and $\mathrm{dvol}|_{S^{n-1}}$ is the Euclidean volume element induced on S^{n-1}. For the term $\partial_{\bar{v}} v_0(p,\epsilon,0)(\bar{v})$ appearing in V we use Lemma 6.4. We have

$$\partial_{\bar{v}} v_0(p,\epsilon,0)(\bar{v}) = \frac{\epsilon^2}{6\,\omega_n}\int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar{v} \,+ \int_{S^{n-1}}[\mathcal{O}(\epsilon^5)+\epsilon^3\,L_p(\bar{v})]$$

We finally obtain

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0}\hat{\lambda}(p,\epsilon,s\bar{v}) = C\,\epsilon^2\int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar{v} \,+ \int_{S^{n-1}}[\mathcal{O}(\epsilon^5)+\epsilon^3\,L_p(\bar{v})]$$

where

$$C = -\frac{c_1^2}{6} - \frac{2\,c_1^2}{12} + \frac{c_1^2}{6} = -\frac{c_1^2}{6}.$$

The proof of the Lemma follows at once. □

Summarizing the results of Lemmas 6.1, 6.2 and 6.3 we obtain the following:

Proposition 6.5. *Let $p \in M$, let ϵ and $\bar{v}$ be small enough. Then:*

$$\begin{aligned}\hat{\lambda}(p,\epsilon,\bar{v}) \;&=\; \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 \\ &\quad -c_1\int_{S^{n-1}} \bar{v}\,H(\bar{v}) \,-\, \frac{c_1^2}{6}\,\epsilon^2\int_{S^{n-1}} \mathring{Ric}_p(\Theta,\Theta)\,\bar{v} \\ &\quad + \int_{S^{n-1}}\left[\mathcal{O}(\epsilon^5)+\epsilon^3\,L_p(\bar{v})+\epsilon^2\,Q_p^{(2)}(\bar{v})+Q_p^{(3)}(\bar{v})\right]\end{aligned}$$

where Θ has been defined in (5), and we agree with the convention about $L_p(v)$, $Q_p^{(2)}(v)$ and $Q_p^{(3)}(v)$ we gave before.

Proof. It suffices to put together the results of Lemmas 6.1, 6.2 and 6.3. □

7. Localisation of the obtained extremal domains

Now we want to find the expansion of the function $\Psi_\epsilon(p)$ in power of ϵ. Recall that

$$\Psi_\epsilon(p) = \hat{\lambda}(p,\epsilon,\bar{v}(p,\epsilon))$$

In order to find such expansion we will relate the first term in the expansion of $\bar{v}(p,\epsilon)$ to the curvature of the manifold at p.

The first term of the expansion of $\bar{v}(p,\epsilon)$ is related to the traceless Ricci curvature at p, as stated by the following:

Proposition 7.1. *We have*

$$\bar{v}(p,\epsilon) = -\frac{c_1}{12\,\alpha_2}\,\mathring{Ric}_p(\Theta,\Theta)\,\epsilon^2 + \mathcal{O}(\epsilon^3) = \frac{n}{12\,(\lambda_1-n)}\,\mathring{Ric}_p(\Theta,\Theta)\,\epsilon^2 + \mathcal{O}(\epsilon^3)$$

where Θ has been defined in (5), and α_2 is the eigenvalue of the operator H defined in Proposition 4.2 associated to the eigenspace V_2.

Proof. Let us recall that

$$F(p,\epsilon,\bar v(p,\epsilon)) + \langle a(p,\epsilon),\cdot\rangle = 0.$$

where

$$\|\bar v(p,\epsilon)\|_{C^{2,\alpha}(S^{n-1})} + \|a(p,\epsilon)\| \leq c\,\epsilon^2$$

Now, because $F(p,\epsilon,0) = \mathcal{O}(\epsilon^2)$ and because $\bar v(p,\epsilon) = \mathcal{O}(\epsilon^2)$, we can write

$$\begin{aligned} F(p,\epsilon,\bar v) &= F(p,0,0) + \partial_\epsilon F(p,0,0)\,\epsilon \\ &\quad + \partial_{\bar v} F(p,0,0)\,\bar v + \frac{1}{2}\partial_\epsilon^2 F(p,0,0)\,\epsilon^2 + \mathcal{O}(\epsilon^3) \\ &= H(\bar v) + \frac{1}{2}\,\partial_\epsilon^2 F(p,0,0)\,\epsilon^2 + \mathcal{O}(\epsilon^3) \end{aligned}$$

In the computation of the mixed derivatives of $\hat\lambda$ in the proof of Lemma 6.3 we have already computed the expansion of $\hat g(\nabla\hat\phi,\hat\nu)$ for $\bar v = 0$, so we directly deduce

$$\begin{aligned} F(p,\epsilon,0) &= \epsilon^2\,\frac{c_1}{12}\left[R_{ij}(p)\,y^i\,y^j - \frac{R_p}{n}\right] + \mathcal{O}(\epsilon^3) \\ &= \epsilon^2\,\frac{c_1}{12}\,\mathring{R}ic_p(\Theta,\Theta) + \mathcal{O}(\epsilon^3). \end{aligned}$$

Then we have

$$\partial_\epsilon^2 F(p,0,0) = \frac{c_1}{6}\,\mathring{R}ic_p(\Theta,\Theta)$$

Writing

$$a = a_p\,\epsilon^2 + \mathcal{O}(\epsilon^3)$$

and

$$\bar v = \bar v_p\,\epsilon^2 + \mathcal{O}(\epsilon^3)$$

and considering the expansion of F, from equation (22) we obtain

$$H(\bar v_p) + \frac{c_1}{12}\,\mathring{R}ic_p(\Theta,\Theta) = -\langle a_p,\cdot\rangle \tag{26}$$

We know that $\bar v$, and hence $\bar v_p$, is L^2-orthogonal to $V_0 \oplus V_1$ (see Propositions 4.3). Observe that $Ric(\Theta,\Theta)$ is $L^2(S^{n-1})$-orthogonal to V_1 since the function $\Theta \to Ric(\Theta,\Theta)$ is invariant when Θ is changed into $-\Theta$ and hence its L^2-projection over elements of the form $g(\Xi,\Theta)$ is 0 for every Ξ. Then $\mathring{R}ic(\Theta,\Theta)$ is $L^2(S^{n-1})$-orthogonal to $V_0 \oplus V_1$. In fact $\mathring{R}ic(\Theta,\Theta)$ is the restriction on S^{n-1} of a homogeneous polynomial of degree 2 which has mean 0, and then it is an eigenfunction for $-\Delta_{S^{n-1}}$ with eigenvalue $2n$. As H preserves the eigenspaces of $-\Delta_{S^{n-1}}$ and his kernel is given by V_1 (see Proposition 4.2), we have that there exists a constant $\alpha_2 \neq 0$ such that

$$H\left(\mathring{R}ic(\Theta,\Theta)\right) = \alpha_2\,\mathring{R}ic(\Theta,\Theta)$$

From (26) we obtain

$$-\langle a_p, \cdot \rangle = H\left(\bar{v}_p + \frac{c_1}{12\alpha_2}\,\mathring{Ric}(\Theta,\Theta)\right)$$

i.e. $\langle a_p, \cdot\rangle$ is in the image of H. But it belongs also to the kernel of H, and then $a_p = 0$ and

$$H\left(v_p + \frac{c_1}{12\alpha_2}\,\mathring{Ric}(\Theta,\Theta)\right) = 0 \tag{27}$$

Now we remark that $\left(v_p + \frac{c_1}{12\alpha_2}\,\mathring{Ric}(\Theta,\Theta)\right)$ is orthogonal to $V_0 \oplus V_1$, and then

$$v_p = -\frac{c_1}{12\alpha_2}\,\mathring{Ric}(\Theta,\Theta) \tag{28}$$

In order to complete the proof of the proposition we use equation (32) and Lemma 8.1 of the Appendix. □

Now we are able to give an expansion for the function $\Psi_\epsilon(p)$ in power of ϵ.

Proposition 7.2. *We have:*

$$\Psi_\epsilon(p) = \lambda_1 + \frac{\hat{\Lambda}_0}{R_p}\,\epsilon^2\,\left(R_p + \mathbf{r}_p\,\epsilon^2\right) + \mathcal{O}(\epsilon^5) \tag{29}$$

where $\hat{\Lambda}_0$ is defined in Lemma 6.1 (note that $\frac{\hat{\Lambda}_0}{R_p}$ is well defined also when $R_p = 0$), and the function $\mathbf{r}$ can be written as

$$\mathbf{r} = K_1\,\|Riem\|^2 + K_2\,\|Ric\|^2 + K_3\,R^2 + K_4\,\Delta_g R$$

for some constants K_i only depending on n.

Proof. Replacing $\bar{v}$ with its expansion given by Proposition 7.1 in the expansion of $\hat{\lambda}$ given by Proposition 6.5, we obtain

$$\begin{aligned}
\Psi_\epsilon(p) &= \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 - c_1 \int_{S^{n-1}} \bar{v}\left(H(\bar{v}) + \frac{c_1}{6}\,\epsilon^2\,\mathring{Ric}_p(\Theta,\Theta)\right) + \mathcal{O}(\epsilon^5)\\
&= \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 - c_1 \int_{S^{n-1}} \bar{v}\left(\alpha_2\,\bar{v} + \frac{c_1}{6}\,\epsilon^2\,\mathring{Ric}_p(\Theta,\Theta)\right) + \mathcal{O}(\epsilon^5)\\
&= \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 + \frac{c_1^3}{144\,\alpha_2}\,\epsilon^4 \int_{S^{n-1}} (\mathring{Ric}_p(\Theta,\Theta))^2 + \mathcal{O}(\epsilon^5)\\
&= \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 + \frac{c_1^3\,\omega_n}{72\,\alpha_2\,n(n+2)}\,\epsilon^4\,\left(\|Ric_p\|^2 - \frac{1}{n}\,R_p^2\right) + \mathcal{O}(\epsilon^5)\\
&= \lambda_1 + \hat{\Lambda}_0\,\epsilon^2 + \hat{\Lambda}\,\epsilon^4 + \frac{\lambda_1}{36(n+2)(n-\lambda_1)}\,\epsilon^4\,\left(\|Ric_p\|^2 - \frac{1}{n}\,R_p^2\right) + \mathcal{O}(\epsilon^5)
\end{aligned}$$

where we used (28) from the second to the third line, the following two geometric formulas

$$\begin{aligned} \int_{S^{n-1}} Ric(\Theta,\Theta) &= \frac{\omega_n}{n}\, R_p \\ \int_{S^{n-1}} (Ric(\Theta,\Theta))^2 &= \frac{\omega_n}{n(n+2)}\,(2\|Ric_p\|^2 + R_p^2)\,, \end{aligned}$$

whose proofs can be found in [13], from the third to the fourth line, and the computation of α_2 given in (32) and Lemma 8.1 to deduce the last line. Define

$$\begin{aligned} \mathbf{r}_p &= R_p\,\hat\Lambda_0^{-1}\left[\hat\Lambda + \frac{\lambda_1}{36(n+2)(n-\lambda_1)}\left(\|Ric_p\|^2 - \frac{1}{n}R_p^2\right)\right] \\ &= R_p\,\hat\Lambda_0^{-1}\left[\Lambda + \lambda_1\left(\frac{2\,W}{\omega_n} - \frac{R_p^2}{36\,n^2\,(n+2)}\right) + \frac{\lambda_1}{36(n+2)(n-\lambda_1)}\left(\|Ric_p\|^2 - \frac{1}{n}\,R_p^2\right)\right] \end{aligned}$$

Recalling the definition of W and Λ given in (11) and (13), we obtain that

$$\mathbf{r}_p = K_1\,\|Riem_p\|^2 + K_2\,\|Ric_p\|^2 + K_3\,R_p^2 + K_4\,(\Delta_g R)_p$$

where

$$\begin{aligned} K_1 &= \frac{1}{n\,(n+2)+2\,\lambda_1}\left(18\,c^2 + \frac{\lambda_1}{10(n+4)}\right) \\ K_2 &= \frac{1}{n\,(n+2)+2\,\lambda_1}\left(\frac{35}{3}\,c^2 + \frac{4\,\lambda_1}{15(n+4)} + \frac{n\lambda_1}{6(\lambda_1-n)}\right) \\ K_3 &= \frac{1}{n\,(n+2)+2\,\lambda_1}\left(\frac{5n-3}{3n}\,c^2 - \frac{\lambda_1}{6(n+4)} + \frac{\lambda_1}{6n} - \frac{\lambda_1}{6(\lambda_1-n)}\right) \\ K_4 &= \frac{1}{n\,(n+2)+2\,\lambda_1}\left(\frac{6}{5}\,c^2 + \frac{3\,\lambda_1}{5(n+4)}\right) \end{aligned} \tag{30}$$

and formula (29) follows at once. The fact that the constants K_i depend only on n comes immediately from the computation of c^2 by Lemma 8.2 in the Appendix:

$$c^2 = \frac{(n+2)\,[2\lambda_1 + n(n-4)]}{12\,\lambda_1\,\omega_n}$$

This completes the proof of the proposition. □

Remark 1. We remark that $K_1 > 0$ in order to justify our discussion about critical point of $\|Riem\|$ for Einstein metrics in the introduction.

Now recalling that

$$\Phi(p,\epsilon) = R_p\,\hat\Lambda_0^{-1}\,\frac{\Psi_\epsilon(p)-\lambda_1}{\epsilon^2} = -\frac{6n(n+2)}{n(n+2)+2\lambda_1}\,\frac{\Psi_\epsilon(p)-\lambda_1}{\epsilon^2}$$

the proof of the second and third part of Theorem 1.1 follows at once.

8. Appendix I : On the first eigenfunction in the unit Euclidean ball

In this Appendix we state and prove some relations between the first eigenfunction and the first eigenvalue of the Dirichlet Laplacian on the unit ball.

Lemma 8.1. *Let*

$$c_1 = \phi_1'(1)$$

where $x \to \phi_1(|x|)$ is the first eigenfunction of the Dirichlet Laplacian on the unit ball, normalized in order to have L^2-norm equal to 1. Then

$$c_1 = -\sqrt{\frac{2\lambda_1}{\omega_n}}$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian on the unit ball.

Proof. Recall that ϕ_1 is the solution of

$$\phi_1'' + \frac{n-1}{r}\phi_1' + \lambda_1\phi_1 = 0$$

with normalization

$$1 = \int_{B_1} \phi_1^2(|x|)\, \mathrm{d}x = \omega_n \int_0^1 (\phi_1)^2\, r^{n-1}\, \mathrm{d}r = -\frac{2\,\omega_n}{n} \int_0^1 \phi_1\, \phi_1'\, r^n\, \mathrm{d}r \tag{31}$$

and

$$\lambda_1 = \int_{B_1} |\nabla\phi_1(|x|)|^2\, \mathrm{d}x = \omega_n \int_0^1 (\phi_1')^2\, r^{n-1}\, \mathrm{d}r$$

Now let us compute

$$\begin{aligned}
(r^n(\phi_1')^2)' &= n\, r^{n-1}\, (\phi_1')^2 + 2r^n\, \phi_1'\, \phi_1'' \\
&= n\, r^{n-1}\, (\phi_1')^2 - 2r^n\, \phi_1' \left(\frac{n-1}{r}\, \phi_1' + \lambda_1\, \phi_1\right) \\
&= (2-n)\, r^{n-1}\, (\phi_1')^2 - 2\lambda_1\, r^n\, \phi_1'\, \phi_1
\end{aligned}$$

Integrating this relation between 0 and 1 we obtain

$$c_1^2 = \frac{2\lambda_1}{\omega_n}$$

The proof of the Lemma follows at once, keeping in mind that c_1 is negative. □

Lemma 8.2. *Let*

$$c^2 = \frac{n+2}{2} \int_0^1 \phi_1^2\, r^{n+1}\, \mathrm{d}r$$

where $x \to \phi_1(|x|)$ is the first eigenfunction of the Dirichlet Laplacian on the unit ball, normalized in order to have L^2-norm equal to 1. Then

$$c^2 = \frac{(n+2)\,[2\lambda_1 + n(n-4)]}{12\,\lambda_1\,\omega_n}$$

where λ_1 is the first eigenvalue of the Dirichelt Laplacian on the unit ball.

Proof. We have

$$\frac{n+2}{2}\int_0^1 \phi_1^2\, r^{n+1}\, \mathrm{d}r = -\int_0^1 \phi_1\, \phi_1'\, r^{n+2}\, \mathrm{d}r$$

Recall also that

$$\phi_1'' + \frac{n-1}{r}\, \phi_1' + \lambda_1 \phi_1 = 0$$

with $\phi_1(1) = 0$, and ϕ_1 is normalized by (31). We first compute

$$\begin{aligned}
(r^{n+2}(\phi_1')^2)' &= (n+2)\, r^{n+1}\, (\phi_1')^2 + 2r^{n+2}\, \phi_1'\, \phi_1'' \\
&= (n+2)\, r^{n+1}\, (\phi_1')^2 - 2r^{n+2}\, \phi_1' \left(\frac{n-1}{r}\, \phi_1' + \lambda_1\, \phi_1\right) \\
&= (4-n)\, r^{n+1}\, (\phi_1')^2 - 2\lambda_1\, r^{n+2}\, \phi_1'\, \phi_1
\end{aligned}$$

Integrating this relation between 0 and 1 we find

$$c_1^2 = (4-n)\int_0^1 r^{n+1}\, (\phi_1')^2 + 2\lambda_1 c^2$$

where $c_1 = \phi_1'(1)$. We now compute

$$\begin{aligned}
(r^{n+1}\, \phi_1\, \phi_1')' &= (n+1)\, r^n\, \phi_1\, \phi_1' + r^{n+1}\, (\phi_1')^2 + r^{n+1}\, \phi_1\, \phi_1'' \\
&= (n+1)\, r^n\, \phi_1\, \phi_1' + r^{n+1}\, (\phi_1')^2 - r^{n+1}\, \phi_1 \left(\frac{n-1}{r}\, \phi_1' + \lambda_1\, \phi_1\right) \\
&= 2r^n\, \phi_1\, \phi_1' + r^{n+1}\, (\phi_1')^2 - \lambda_1\, r^{n+1}\, \phi_1^2 \\
&= (r^n\, \phi_1^2)' - n\, r^{n-1}\, \phi_1^2 + r^{n+1}\, (\phi_1')^2 - \lambda_1\, r^{n+1}\, \phi_1^2
\end{aligned}$$

Integrating this relation between 0 and 1 we find

$$0 = -n\, (\omega_n)^{-1} + \int_0^1 r^{n+1}\, (\phi_1')^2 - \lambda_1 \frac{2}{n+2} c^2\, .$$

Thus we have at the end

$$c^2 = \frac{n+2}{12\lambda_1}\left[c_1^2 + \frac{n(n-4)}{\omega_n}\right]\, .$$

The proof of the Lemma follows at once from Lemma 8.1. $\square$

9. Appendix II: The second eigenvalue of the operator H

Here we compute the eigenvalue α_2 of the operator H associated to the eigenspace V_2. When w is an homogeneous polynomial harmonic of degree 2 (abusively identified with its restriction to the unit sphere) we have $\Delta_{S^{n-1}} w = -\mu_2 w = -2n\, w$ and $H(w) = \alpha_2\, w$. We recall that

$$H(w) = (\partial_r \psi)|_{\partial B_1} + c_2\, w = (\partial_r \psi)|_{\partial B_1} - (n-1)\, c_1\, w$$

where ψ is the solution of

$$\left\{ \begin{array}{rclll} \Delta \psi + \lambda_1\, \psi & = & 0 & \text{in} & B_1 \\[3mm] \psi & = & -c_1\, w & \text{on} & \partial B_1 \end{array} \right.$$

which is $L^2(B_1)$-orthogonal to ϕ_1. Decomposing ψ in spherical harmonics, we see that $\psi(r,\theta) = b_2(r)\, w(\theta)$ where b_2 is the solution defined at 0 of

$$\left\{ \begin{array}{l} r^2\, b'' + (n-1)\, r\, b' + (r^2\, \lambda_1 - 2n)\, b = 0 \quad \text{in} \quad (0,1) \\[3mm] b(1) = -c_1 = -\phi_1'(1) \end{array} \right.$$

From the definition of H, we see that

$$\alpha_2 = b_2'(1) + \phi_1''(1) = b_2'(1) + c_2 = b_2'(1) - (n-1)\, c_1$$

so we have to compute $b_2'(1)$. Let us verify that

$$b_2(r) = -\left(\frac{\lambda_1}{n}\, \phi_1 + \frac{1}{r}\, \phi_1' \right)$$

is the desired solution. Recall that

$$\phi_1'' + \frac{n-1}{r}\, \phi_1' + \lambda_1 \phi_1 = 0,$$

thus

$$(\phi_1')'' + \frac{n-1}{r}\, (\phi_1')' + \lambda_1 \phi_1' = \frac{n-1}{r^2}\, \phi_1'$$

Now

$$b_2' = -\left(\frac{\lambda_1}{n}\, \phi_1' + \frac{1}{r}\, \phi_1'' \right) + \frac{1}{r^2}\, \phi_1'$$

and

$$b_2'' = -\left(\frac{\lambda_1}{n}\, \phi_1'' + \frac{1}{r}\, \phi_1''' \right) + \frac{2}{r^2}\, \phi_1'' - \frac{2}{r^3}\, \phi_1'$$

so

$$\begin{aligned} b_2'' + \frac{n-1}{r}\, b_2' + \lambda_1\, b_2 &= -\frac{1}{r}\,\frac{n-1}{r^2}\,\phi_1' + \frac{n-1}{r}\,\frac{1}{r^2}\,\phi_1' + 2\,\frac{1}{r^2}\,\phi_1'' - 2\,\frac{1}{r^3}\,\phi_1' \\ &= -2\,\frac{1}{r^2}\left(\frac{n-1}{r}\,\phi_1' + \lambda_1\,\phi_1\right) - 2\,\frac{1}{r^3}\,\phi_1' \\ &= -\frac{2n}{r^2}\left(\frac{\lambda_1}{n}\,\phi_1 + \frac{1}{r}\,\phi_1'\right) \\ &= \frac{2n}{r^2}\, b_2 \end{aligned}$$

And of course $b_2(1) = -c_1$, so this is the desired solution. Finally we have

$$b_2'(1) = \frac{n^2 - \lambda_1}{n} c_1$$

and

$$\alpha_2 = \frac{n-\lambda_1}{n} c_1 = \frac{\lambda_1 - n}{n}\sqrt{\frac{2\lambda_1}{\omega_n}} > 0. \tag{32}$$

10. Appendix III : Differentiating with respect to the domain

In this Appendix we recall a useful result that allows to derivate the integral of a function with respect to a parameter t that appears in the function and also in the domain of integration. The proof of such result can be found in [7], page 14.

Proposition 10.1. *Let Ω a smooth bounded domain of $\mathbb{R}^n$ and*

$$h : (-r, r) \times \Omega \to \mathbb{R}^n$$

a smooth function, where r is a positive constant, such that $h(0, p) = p$ for all $p \in \Omega$. Let

$$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

a smooth function. Let $\Omega_t = h(t, \Omega_0)$, $V(t, h(t, p)) = \frac{\partial h}{\partial t}(t, p)$ and $N(t, q)$ the unit outward normal at $q \in \partial\Omega_t$. Then

$$\frac{\partial}{\partial t}\int_{\Omega_t} f = \int_{\Omega_t} \frac{\partial f}{\partial t}\, dx + \int_{\partial\Omega_t} f\, \langle V, N\rangle\, ds \tag{33}$$

and

$$\frac{\partial}{\partial t}\int_{\partial\Omega_t} f\, ds = \int_{\partial\Omega_t}\left(\frac{\partial f}{\partial t} + \langle V, N\rangle\, \langle \nabla_x f, N\rangle + H\,\langle V, N\rangle\, f\right) ds \tag{34}$$

where $\langle\cdot,\cdot\rangle$ denote the scalar product in $\mathbb{R}^n$, s denote the area element of $\partial\Omega_t$ and H is the mean curvature of $\partial\Omega_t$.

Remark 2. Although we do not need it here, we mention that this proposition can easily be proven also for domains in a Riemannian manifold.

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Erwann Delay, Laboratoire de Mathématiques d'Avignon, Faculté des Sciences, 33 rue Louis Pasteur, F-84000 Avignon, France
E-mail address: erwann.delay@univ-avignon.fr
URL: http://www.univ-avignon.fr/en/research/annuaire-chercheurs/membrestruc/personnel/delay-erwann-1.html

Pieralberto Sicbaldi, Institut de Mathématiques de Marseille, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France
E-mail address: pieralberto.sicbaldi@univ-amu.fr
URL: http://www.latp.univ-mrs.fr/~sicbaldi